

Control systems on flag manifolds and their chain control sets

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Abstract

In this paper we shown that the chain control sets of induced systems on flag manifolds coincides with their analogous one defined via semigroup actions. Consequently, any chain control set of the system contains a control set with nonempty interior and, if the number of the control sets with nonempty interior coincides with the number of the chain control sets, then the closure of any control set with nonempty interior is a chain control set.

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1 Introduction

A right-invariant control system on a connected Lie group G is the family of differential equations given by

$$\dot{g}(t) = X(g(t)) + \sum_{j=1}^m u_j(t) Y^j(g(t)), \quad u \in \mathcal{U} \quad (\Sigma)$$

where X_0, X_1, \dots, X_m are right invariant vector fields and $\mathcal{U} \subset \mathbb{L}_\infty(\mathbb{R}, U \subset \mathbb{R}^m)$ where $U \subset \mathbb{R}^m$ is a compact and convex set with $0 \in \text{int } U$.

In particular, when G is a noncompact semisimple Lie group one can study induced systems on its flag manifolds. That is, if Σ is a right-invariant system on G , we have induced affine control systems on every flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ given by

$$\dot{x}(t) = f_0^\Theta(x(t)) + \sum_{j=1}^m u_j(t) f_j^\Theta(x(t)), \quad u \in \mathcal{U} \quad (\Sigma_\Theta).$$

where $f_i^\Theta = (d\pi_\Theta)X_i$ for $i = 0, \dots, m$ and $\pi_\Theta : G \rightarrow \mathbb{F}_\Theta$ is the canonical projection.

Studying systems on flag manifolds is desirable for at least two reasons: First, flag manifolds includes, in particular, spheres, projective spaces and Grassmannian spaces. Secondly, right-invariant systems appears in many important applications in engineering and physics such as the orientation of rigid bodies and optimal problems (see [3] and [7]).

In order to understand the dynamical behavior of Σ_Θ one have to analyze control and chain control sets of the system. There is a way to analyze the chain control sets of Σ_Θ through the control flow ϕ^Θ induced by the system on the fiber bundle

$$\phi^\Theta : \mathbb{R} \times \mathcal{U} \times \mathbb{F}_\Theta \rightarrow \mathcal{U} \times \mathbb{F}_\Theta.$$

In fact, since \mathbb{F}_Θ is a compact manifold, Theorem 4.3.11 of [4] implies that there exists a bijection between the Morse components of the control flow ϕ^Θ and the chain control sets of the system Σ_Θ . Also, by [2], all the Morse sets of the flow ϕ^Θ are given fiberwise as fixed points in \mathbb{F}_Θ of regular elements of G .

On the other hand, since the positive orbit $\mathcal{S} := \mathcal{O}^+(1)$ is a semigroup we have the notion of control and chain control sets associated with \mathcal{S} (see for instance [1], [10] and [12]). In particular, any effective chain control set of \mathcal{S} contains a control set with nonempty interior.

In this paper we show that both notion actually agree, that is, the chain control sets of the induced systems Σ_Θ and the effective chain control sets associated with \mathcal{S} are actually the same. That implies, in particular, that any given chain control sets of Σ_Θ contains a control set with nonempty interior. Moreover, if the number of chain control sets coincides with the number of the control sets

with nonempty interior then any chain control set is the closure of the control set that it contains.

The paper is structured as follows: In Section 2 we give some preliminaries on control and Lie theory and enunciate the some results that will be needed. In Section 3 we state and prove our main result, showing that both notions of chain control sets coincides. As a consequence, we prove that if the number of chain control sets and of control sets with nonempty interior coincide, then any chain control set is the closure of the only control set with nonempty interior that it contains.

2 Preliminaries

In this section we introduce all the control theory concepts needed in order to present our main result.

2.1 Control Theory

Let M be a Riemannian differentiable manifold and consider the admissible class of control $\mathcal{U} \subset \mathbb{L}_\infty(\mathbb{R}, U \subset \mathbb{R}^m)$. A control affine system on M is the family of differential equations

$$\dot{x}(t) = f_0(x(t)) + \sum_{j=1}^m u_j(t) f_j(x(t)), \quad x(t) \in M. \quad (1)$$

where f_0, f_1, \dots, f_m are smooth vector fields on M . The set \mathcal{U} is a compact metrizable space and the shift flow

$$\theta : \mathbb{R} \times \mathcal{U} \rightarrow \mathcal{U}, \quad (t, u) \mapsto \theta_t u := u(\cdot + t)$$

is a continuous dynamical system, that is chain transitive (see [4]).

For $u \in \mathcal{U}$ fixed, we denote by $\varphi(\cdot, x, u)$ the unique solution of (1) with $\varphi(0, x, u) = x$. If the vector fields f_0, \dots, f_m are elements of \mathcal{C}^∞ , then φ has also the same class of differentiability respect to M and the corresponding partial derivatives depend continuously on $(t, x, u) \in \mathbb{R} \times M \times \mathcal{U}$ (see Thm. 1.1 of [8]).

If we assume that all the solutions determined by \mathcal{U} are defined in the whole time axis¹, we have the map

$$\varphi : \mathbb{R} \times M \times \mathcal{U} \rightarrow M, \quad (t, x, u) \mapsto \varphi(t, x, u),$$

called the transition map of the system. The transition map together with the shift flow determines a skew-product flow

$$\phi : \mathbb{R} \times \mathcal{U} \times M \rightarrow \mathcal{U} \times M, \quad (t, x, u) \mapsto \phi_t(u, x) = (\theta_t u, \varphi(t, x, u)),$$

¹This condition is in general restrictive, but for the class of affine control systems that we will study it holds.

called the control flow of the system (see Sec. 4.3 of [4]).

The sets

$$\begin{aligned}\mathcal{O}_\tau^+(x) &:= \{\varphi(\tau, x, u) : u \in \mathcal{U}\}, \\ \mathcal{O}_{\leq \tau}^+(x) &:= \bigcup_{t \in [0, \tau]} \mathcal{O}_t^+(x) \quad \text{and} \quad \mathcal{O}^+(x) := \bigcup_{\tau > 0} \mathcal{O}_\tau^+(x).\end{aligned}$$

are the set of reachable points from $x \in M$ at time $\tau > 0$, the set of reachable points from x up to time τ , and the **positive orbit of x** , respectively. Analogously we define the sets $\mathcal{O}_\tau^-(x)$, $\mathcal{O}_{\leq \tau}^-(x)$ and $\mathcal{O}^-(x)$ to be the set of the points controllable to x in time $\tau > 0$, the set of points controllable to x up to time τ and the **negative orbit of x** , respectively.

The affine control system (1) is called locally accessible at x if for all $\tau > 0$ the sets $\mathcal{O}_{\leq \tau}^+(x)$ and $\mathcal{O}_{\leq \tau}^-(x)$ have nonempty interiors. It is called **locally accessible** if it is locally accessible at every point $x \in M$. Under the assumption of locally accessibility, it holds that $\text{int } \mathcal{O}^+(x)$ is a dense subset of $\mathcal{O}^+(x)$, for any $x \in M$ (see Lemma 1.2 of [8]).

2.1 Definition: A control set of the system (1) is a subset $D \subset M$ satisfying:

- (i) for each $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}_+, x, u) \subset D$
- (ii) $D \subset \text{cl } \mathcal{O}^+(x)$ for all $x \in D$
- (iii) D is maximal with respect to the set inclusion and properties (i) and (ii).

In [4], Proposition 3.2.4, it is shown that a subset D with nonempty interior, that is maximal with the property (ii) of the above definition is also a control set. This fact shows, in particular, that not any control set of (1) has to have nonempty interior.

Let us now fix a metric ϱ on M . For $x, y \in M$ and $\varepsilon, \tau > 0$, a **controlled (ε, τ) -chain** from x to y is given by an integer $n \in \mathbb{N}$ and the existence of three finite sequences

$$x_0, \dots, x_n \in M, \quad u_0, \dots, u_{n-1} \in \mathcal{U}, \quad t_0, \dots, t_{n-1} \geq \tau$$

such that

$$x = x_0, \quad y = x_n \quad \text{and} \quad \varrho(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon, \quad \text{for } i = 0, \dots, n-1.$$

2.2 Definition: A set $E \subset M$ is called a **chain control set of (1)** if it satisfies the following properties:

- i) for each $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset E$
- ii) For all $x, y \in E$ and $\varepsilon, \tau > 0$ there exists an (ε, τ) -chain from x to y in M

- iii) E is maximal with respect to the set inclusion and the properties (i) and (ii) above.

We say that a set E satisfying condition (ii) in the above definition is **controlled chain transitive**.

2.3 Remark: From the general theory, every chain control set of the control system (1) is closed, but this is not necessarily true for control sets. Moreover, every control set with nonempty interior is contained in a chain control set if local accessibility holds (see Sec. 4.3 of [4]).

2.2 Lie Theory and Flag Manifolds

In order to state and prove our main result, in this section we give some facts about semisimple Lie groups and their induced flag manifolds.

2.2.1 Semisimple Lie Groups

Let G be a connected non-compact semisimple Lie group G with finite center and Lie algebra \mathfrak{g} . Fix a Cartan involution $\zeta : \mathfrak{g} \rightarrow \mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal Abelian subalgebra and $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber. Let us denote by Π , Π^+ and $\Pi^- := -\Pi^+$ the **set of roots**, the **set of positive roots** and **set of negative roots**, respectively, associated with \mathfrak{a}^+ . The **Iwasawa decomposition** of the Lie algebra \mathfrak{g} , associated with the above choices, reads as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^\pm \text{ where } \mathfrak{n}^\pm := \sum_{\alpha \in \Pi^\pm} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g}, [H, X] = \alpha(H)X, \text{ for all } H \in \mathfrak{a}\}$.

Let K , A and N^\pm be the connected Lie subgroups of G with Lie algebras \mathfrak{k} , \mathfrak{a} and \mathfrak{n}^\pm , respectively. The Iwasawa decomposition of the Lie group G is given by $G = KAN^\pm$. The Weyl group of \mathfrak{g} associated to $\alpha \in \Pi$ is the finite group generated by the reflections on the hyperplane $\ker \alpha$. Alternatively, the Weyl group can be obtained by the quotient M^*/M , where M^* and M are respectively the normalizer and the centralizer of \mathfrak{a} in K .

We denote by $\Lambda \subset \Pi^+$ the set of the positive roots which cannot be written as linear combinations of other positive roots. The Weyl groups coincides with the group generated by the reflections associated to $\alpha \in \Lambda$. There is only one involutive element $w_0 \in \mathcal{W}$ such that $w_0\Pi^+ = \Pi^-$.

For $\Theta \subset \Lambda$ the parabolic subalgebra of type Θ is $\mathfrak{p} := \mathfrak{n}^-(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ where \mathfrak{m} is the Lie algebra of M and $\mathfrak{n}^-(\Theta)$ is the sum of the eigenspaces \mathfrak{g}_α when $\alpha \in \langle \Theta \rangle \cap \Pi^-$, where $\langle \Theta \rangle \subset \Pi$ is the set of roots given as linear combination of the simple roots in Θ . The corresponding parabolic subgroup P_Θ is the

normalizer of \mathfrak{p}_Θ in G . The flag manifold \mathbb{F}_Θ is the orbit $\mathbb{F}_\Theta := \text{Ad}(G)\mathfrak{p}_\Theta$ on a Grassmannian manifold. It is identified with the homogeneous space G/P_Θ . When $\Theta = \emptyset$, the sets $\mathfrak{p} := \mathfrak{p}_\emptyset$ and $\mathbb{F} := \mathbb{F}_\emptyset$ are called the minimal parabolic subalgebra and the maximal flag manifold, respectively.

An element of \mathfrak{g} of the form $Y = \text{Ad}(g)H$ with $g \in G$ and $H \in \text{cl } \mathfrak{a}^+$ is called a **split element**. When $H \in \mathfrak{a}^+$ we say that Y is a **split regular** element. The flow induced by a split element $H \in \text{cl } \mathfrak{a}^+$ on \mathbb{F}_Θ , is given by

$$(t, \text{Ad}(g)\mathfrak{p}) \mapsto \text{Ad}(e^{tH}g)\mathfrak{p}_\Theta.$$

It turns out that the associated vector field is a gradient vector field with respect to an appropriate Riemannian metric on \mathbb{F}_Θ .

The connected components of the fixed point set of this flow are given by

$$\text{fix}_\Theta(H, w) = Z_H \cdot wb_\Theta = K_H \cdot wb_\Theta, \quad w \in \mathcal{W}.$$

Here b_Θ is the origin of \mathbb{F}_Θ , Z_H is the centralizer of H in G and $K_H = Z_H \cap K$. The sets $\text{fix}_\Theta(H, w)$ are in bijection with the double coset space $\mathcal{W}_H \backslash \mathcal{W} / \mathcal{W}_\Theta$ where \mathcal{W}_H and \mathcal{W}_Θ are the subgroups of the Weyl group generated, respectively, by the simple roots in $\Theta(H) := \{\alpha \in \Lambda, \alpha(H) = 0\}$ and in Θ .

Each component $\text{fix}_\Theta(H, w)$ is a compact connected submanifold of \mathbb{F}_Θ . Moreover, for $Y = \text{Ad}(g)H$ with $H \in \text{cl } \mathfrak{a}^+$ we get that

$$\text{fix}_\Theta(Y, w) = \text{fix}_\Theta(\text{Ad}(g)H, w) = g \cdot \text{fix}_\Theta(H, w).$$

When H is a split regular element the set $\text{fix}_\Theta(H, w)$ reduces to the point wb_Θ and consequently $\text{fix}_\Theta(Y, w) = gwb_\Theta$.

2.2.2 Semigroups

Here we introduce the notion of control and chain control sets given by a semigroup. Let \mathcal{S} be a semigroup of a Lie group G and M a space provided with a G -transitive action. A \mathcal{S} -control set on M is a subset $D \subset M$ satisfying

- (i) $\text{int } D \neq \emptyset$
- (ii) $D \subset \text{cl}(\mathcal{S}x)$ for all $x \in D$
- (iii) D is maximal (w.r.t. set inclusion) with the properties (i) and (ii).

A \mathcal{S} -control set D is said to be **effective** if the set $D_0 = \{x \in D, x \in (\text{int } \mathcal{S})x\}$ is nonempty. The subset D_0 is said to be the **core** of D .

For semigroups with nonempty interior of a semisimple Lie group G we have the following result from [12].

2.4 Theorem: For any $w \in \mathcal{W}$ there is an effective \mathcal{S} -control set $D_\Theta(w) \subset \mathbb{F}_\Theta$ whose core is given by

$$D_\Theta(w)_0 = \{\text{fix}_\Theta(Y, w); \quad Y \text{ is split regular and } e^Y \in \text{int } \mathcal{S}\}.$$

Moreover, $D_\Theta(1)$ is the only invariant \mathcal{S} -control set, $D_\Theta(w_0)$ the only invariant \mathcal{S}^{-1} -control set and any effective \mathcal{S} -control set in \mathbb{F}_Θ is of the form $D_\Theta(w)$ for some $w \in \mathcal{W}$.

It follows from the above that there exists a unique $\Theta(\mathcal{S}) \subset \Lambda$ such that the set $\{w \in \mathcal{W} : D(w) = D(1)\} \subset \mathcal{W}$ coincides with the subgroup $\mathcal{W}_{\Theta(\mathcal{S})}$ of \mathcal{W} . Moreover,

$$\mathcal{W}_{\Theta(\mathcal{S})}w_1\mathcal{W}_\Theta = \mathcal{W}_{\Theta(\mathcal{S})}w_2\mathcal{W}_\Theta \Leftrightarrow D_\Theta(w_1) = D_\Theta(w_2)$$

and the number of effective control sets in \mathbb{F}_Θ is equal to the number of elements in the coset space $\mathcal{W}_{\Theta(\mathcal{S})} \backslash \mathcal{W}/\mathcal{W}_\Theta$. The subset $\Theta(\mathcal{S})$ is called the **flag type of the semigroup \mathcal{S}** .

We also have a concept of chain control sets associated with a semigroup as follows: Let \mathcal{F} be a family of subsets of \mathcal{S} . For $x, y \in M$, $\varepsilon > 0$ and $A \in \mathcal{F}$ a $(\mathcal{S}, \varepsilon, A)$ -**chain** from x to y is given by an integer $n \in \mathbb{N}$, $x_0, \dots, x_n \in M$, $g_0, \dots, g_{n-1} \in A$ such that $x_0 = x$, $x_n = y$ and

$$\varrho(g_i x_i, x_{i+1}) < \varepsilon, \quad \text{for } i = 0, \dots, n-1.$$

A subset $E_\mathcal{F} \subset M$ is called a **\mathcal{F} -chain control set** of \mathcal{S} if it satisfies

- (i) $\text{int } E_\mathcal{F} \neq \emptyset$.
- (ii) For all $x, y \in E_\mathcal{F}$, $\varepsilon > 0$ and $A \in \mathcal{F}$ there exists an $(\mathcal{S}, \varepsilon, A)$ -chain from x to y .
- (iii) $E_\mathcal{F}$ is maximal (w.r.t. set inclusion) with the properties (i) and (ii).

A \mathcal{F} -chain control set is said to be **effective** if it contains an effective \mathcal{S} -control set.

When G is a noncompact semisimple Lie group with finite center, we have the following refined result (see [1], Proposition 4.7).

2.5 Proposition: For each $w \in \mathcal{W}$ there exists an effective \mathcal{F} -chain control set $E_{\mathcal{F}, \Theta}(w)$ for \mathcal{S} on \mathbb{F}_Θ such that $D_\Theta(w) \subset E_{\mathcal{F}, \Theta}(w)$.

Moreover, the set $\mathcal{W}_\mathcal{F}(\mathcal{S}) = \{w \in \mathcal{W} : E_\mathcal{F}(w) = E_\mathcal{F}(1)\}$ is a subgroup of \mathcal{W} and

$$\mathcal{W}_\mathcal{F}(\mathcal{S})w_1\mathcal{W}_\Theta = \mathcal{W}_\mathcal{F}(\mathcal{S})w_2\mathcal{W}_\Theta \Leftrightarrow E_{\mathcal{F}, \Theta}(w_1) = E_{\mathcal{F}, \Theta}(w_2).$$

The unique subset $\Theta(\mathcal{F}) \subset \Lambda$ such that $\mathcal{W}_{\Theta(\mathcal{F})} = \mathcal{W}_{\mathcal{F}}(S)$ is called the **\mathcal{F} -flag type of the semigroup \mathcal{S}** . The number of chain control sets in \mathbb{F}_{Θ} is equal to the number of elements in the coset space $\mathcal{W}_{\mathcal{F}}(S) \setminus \mathcal{W}/\mathcal{W}_{\Theta}$. When there is no change of confusion, we denote only by $E_{\Theta}(w)$ the above \mathcal{F} -chain control sets on \mathbb{F}_{Θ} .

By Proposition 2.5 above, we have that $\Theta(\mathcal{S}) \subset \Theta(\mathcal{F})$ for any family of subsets \mathcal{F} of \mathcal{S} .

2.6 Remark: The above definitions of \mathcal{S} -control sets and \mathcal{F} -chain control sets differs from their analogous ones for control systems, mainly by the assumption that both, \mathcal{S} -control sets and \mathcal{F} -chain control sets, have nonempty interior which is not necessarily true for control systems.

2.2.3 Flow on flag bundles

Let X be a compact metric space and $\phi : \mathbb{R} \times X \rightarrow X$, $(t, x) \mapsto \phi_t(x)$ be a continuous flow. A compact subset $C \subset X$ is called isolated invariant if $\phi_t(C) \subset C$ for all $t \in \mathbb{R}$ and if there is a neighborhood V of C that satisfy

$$\phi_t(x) \in V \quad \text{for all } t \in \mathbb{R} \Rightarrow x \in C.$$

A **Morse decomposition** of ϕ is a finite collection $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$ of nonempty pairwise disjoint isolated invariant compact sets satisfying

- (A) for all $x \in X$ the ω^* and ω -limit sets $\omega^*(x)$ and $\omega(x)$, respectively, are contained in $\bigcup_{i=1}^n \mathcal{M}_i$.
- (B) if there are $\mathcal{M}_{j_0}, \dots, \mathcal{M}_{j_l}$ and $x_1, \dots, x_l \in X \setminus \bigcup_{i=1}^n \mathcal{M}_i$ with

$$\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}} \quad \text{and} \quad \omega(x_i) \subset \mathcal{M}_{j_i}$$

for $i = 1, \dots, l$ then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements in a Morse decomposition are called **Morse sets**.

Let G be a connected semisimple noncompact Lie group with finite center and $\pi : Q \rightarrow X$ a G -principal bundle, where X is a compact metric space. The Lie group G acts continuously on from the right on Q , this action preserves the fibers, and is free and transitive on each fiber. The **flag bundle** $\mathbb{E}_{\Theta} = Q \times_G \mathbb{F}_{\Theta}$ with typical fiber \mathbb{F}_{Θ} is given by $(Q \times \mathbb{F}_{\Theta})/\sim$, where $(q_1, b_1) \sim (q_2, b_2)$ iff there exists $g \in G$ with $q_1 = q_2 \cdot g$ and $b_1 = g^{-1} \cdot b_2$.

Now let $\phi_t : Q \rightarrow Q$ be a flow of automorphisms, i.e., $\phi_t(q \cdot g) = \phi_t(q) \cdot g$, and assume that the induced flow on X is chain transitive. For the induce flow $\phi_t^{\Theta} : \mathbb{E}_{\Theta} \rightarrow \mathbb{E}_{\Theta}$ we have the following result (see Thm. 9.11 of [2], Thm. 5.2 of [11]).

2.7 Theorem: *There exist $H_\phi \in \text{cl}(\mathfrak{a}^+)$ and a continuous ϕ -invariant map*

$$h : Q \rightarrow \text{Ad}(G)H_\phi, \quad h(\phi_t(q)) = h(q),$$

satisfying $h(q \cdot g) = \text{Ad}(g^{-1})h(q)$, $q \in Q$, $g \in G$ and such that the induced flow on \mathbb{E}_Θ admits a finest Morse decomposition whose elements are given fiberwise by

$$\mathcal{M}_\Theta(w)_{\pi(q)} = q \cdot \text{fix}_\Theta(h(q), w).$$

The subset $\Theta(\phi) = \Theta(H_\phi)$ is called the **flag type of the flow** ϕ . The number of the Morse sets in \mathbb{E}_Θ is equal to the number of elements in the coset space $\mathcal{W}_{\Theta(\phi)} \backslash \mathcal{W} / \mathcal{W}_\Theta$.

3 The main result

Here we show that for induced control systems on flag manifolds, both concepts of control and chain control sets above coincide implying that one could analyze the behaviour of invariant control systems via semigroup theory. In particular, the main theorem implies that any chain control sets of a locally accessible induce system contains a control set with nonempty interior, which is in general not necessarily true (see Example 3.4.2 of [4]). Moreover, when $\Theta(\mathcal{S}) = \Theta(\phi)$ every chain control set is the closure of the only control set with nonempty interior that it contains.

Let G be a noncompact connected semisimple Lie group with finite center and consider the right-invariant system Σ on G . Moreover, assume that Σ is a locally accessible right-invariant system on G , and therefore, that all the induced system Σ_Θ are also locally accessible. From here on \mathcal{S} will stand for the semigroup given by $\mathcal{O}^+(1)$, which by the local accessibility condition has nonempty interior in G .

3.1 Proposition: *A subset $D \subset \mathbb{F}_\Theta$ is an effective \mathcal{S} -control set if and only if it is a control set of Σ_Θ with nonempty interior.*

Proof: By Proposition 3.2.4 of [4] and the invariance of Σ we have that any effective \mathcal{S} -control set in \mathbb{F}_Θ is in fact a control set of Σ_Θ with nonempty interior.

Reciprocally, if $D \subset \mathbb{F}_\Theta$ is a control set with nonempty interior of Σ_Θ , we have by invariance that D is a \mathcal{S} -control set and we only have to show that it is effective.

By the local accessibility assumption, we have that $\text{int } D$ is dense in D (see Lemma 3.2.13 of [4]). Moreover, since $\text{int } \mathcal{S}$ is dense in \mathcal{S} and $1 \in \mathcal{S}$ we have that

$$(\text{int } \mathcal{S})^{-1} \cdot x \cap D \neq \emptyset, \quad \text{for any } x \in \text{int } D.$$

Therefore, by property (ii) in the definition of control sets, $(\text{int } \mathcal{S})x \cap (\text{int } \mathcal{S})x \neq \emptyset$ for any $x \in \text{int } D$ which implies that D is in fact effective concluding the proof. \square

3.2 Remark: We should note that the above result does not use the fact that the Lie group G is semisimple, which implies that it remains true for an induced system on an arbitrary homogeneous space.

Now we turn out to the problem of the chain control sets of Σ_Θ . Consider the G -principal bundle $\pi : \mathcal{U} \times G \rightarrow \mathcal{U}$, where π is the projection on the first factor and G acts from the right on $\mathcal{U} \times G$, trivially, as $(u, g) \cdot h = (u, gh)$. The invariance of Σ implies that $\varphi(t, gh, u) = \varphi(t, g, u)h$ which implies that the control flow ϕ_t is a flow of automorphisms on $\mathcal{U} \times G$. Moreover, the induced flow on \mathcal{U} is the shift flow, that is chain transitive. Therefore, Theorem 2.7 can be applied in order to obtain the Morse sets of the induced control flows ϕ^Θ for any $\Theta \subset \Lambda$.

Let \mathcal{F} be the family of subset of \mathcal{S} given by

$$\mathcal{F} := \left\{ \bigcup_{t > \tau} \mathcal{O}_t^+(e), \tau > 0 \right\}.$$

Now we prove our main result.

3.3 Theorem: *A subset $E \subset \mathbb{F}_\Theta$ is a \mathcal{F} -chain control set if and only if it is a chain control set of Σ_Θ .*

Proof: For a given subset E of \mathbb{F}_Θ the lift of E is given by

$$\mathcal{E} = \{(u, x) \in \mathcal{U} \times \mathbb{F}_\Theta; \varphi^\Theta(\mathbb{R}, x, u) \subset E\}.$$

By Theorem 4.3.11 of [4] there exists a bijection between the Morse sets of ϕ^Θ and the chain control sets of the system Σ_Θ given by:

If \mathcal{M}_Θ is a Morse set of ϕ^Θ , then $\pi_2(\mathcal{M}_\Theta)$ is a chain control set of Σ_Θ , where $\pi_2 : \mathcal{U} \times \mathbb{F}_\Theta \rightarrow \mathbb{F}_\Theta$ is the projection on the second factor. Reciprocally, if E is a chain control set of Σ_Θ its lift \mathcal{E} is a Morse set of ϕ^Θ .

We divide the rest of our proof in three steps:

Step 1: If $E_\mathcal{F}$ is a \mathcal{F} -chain control set in \mathbb{F}_Θ then its lift $\mathcal{E}_\mathcal{F}$ is contained in a Morse set of ϕ^Θ .

By condition (ii) in the definition of \mathcal{F} -chain control sets and the right-invariance of Σ we have that $E_\mathcal{F}$ is controlled chain transitive. By Theorem 4.3.11 one gets, in particular, that its the lift $\mathcal{E}_\mathcal{F}$ is chain transitive for the flow ϕ^Θ . Moreover, since $\mathcal{E}_\mathcal{F}$ is certainly ϕ^Θ -invariant, Theorem B.2.26 of [4] implies that $\mathcal{E}_\mathcal{F}$ is contained in some Morse set, since the Morse sets are the maximal ϕ^Θ -invariant chain transitive subsets of $\mathcal{U} \times \mathbb{F}_\Theta$.

Step 2: Any effective \mathcal{F} -chain control set in \mathbb{F}_Θ is a chain control set of Σ_Θ .

Let E be an effective \mathcal{F} -chain control in \mathbb{F}_Θ and $D \subset E$ an effective control set. By Proposition 3.1, D is a control set of Σ_Θ with nonempty interior. By

Corollary 4.3.12 of [4] there exists a chain control set E' of Σ_Θ such that $D \subset E'$. By the very definition of \mathcal{F} -chain control sets and the fact that $\text{int } D \neq \emptyset$ we get that E' is an effective \mathcal{F} -chain control set in \mathbb{F}_Θ which by maximality implies that $E' \subset E$. By the previous item the inclusion $E \subset E'$ must always happens and so $E = E'$ is a chain control set of Σ_Θ .

Step 3: Any chain control set of Σ_Θ is an effective \mathcal{F} -chain control set.

Let E be a chain control set of Σ_Θ . In order to show that E_Θ is a \mathcal{F} -chain control set, it is enough to show that E contains an effective \mathcal{S} -control set.

Define $\mathbf{h} : \mathcal{U} \rightarrow \text{Ad}(G)H_\phi$ by $\mathbf{h}(u) := h(u, 1)$, where h is the map given by Theorem 2.7 applied to the control flow ϕ on the principal bundle $\mathcal{U} \times G$. The map \mathbf{h} is continuous and we have that $\text{Ad}(\varphi(t, e, u))\mathbf{h}(u) = \mathbf{h}(\theta_t u)$. Moreover, the relation between the Morse sets and the chain control sets together with Theorem 2.7 give us that

$$E = \bigcup_{u \in \mathcal{U}} \text{fix}_\Theta(\mathbf{h}(u), w), \quad \text{for some } w \in \mathcal{W}.$$

Let $g \in \text{int } \mathcal{S}$ and consider $u \in \mathcal{U}$ and $\tau > 0$ such that $g = \varphi(\tau, e, u)$. By extending periodically u to a τ -periodic control function $u^* \in \mathcal{U}$ we have that

$$g^n = \varphi(\tau, 1, u)^n = \varphi(n\tau, 1, u^*) \in \text{int } \mathcal{S}.$$

Therefore,

$$g \text{fix}_\Theta(\mathbf{h}(u^*), w) = \text{fix}_\Theta(\text{Ad}(g)\mathbf{h}(u^*), w) = \text{fix}_\Theta(\mathbf{h}(\theta_\tau u^*), w) = \text{fix}_\Theta(\mathbf{h}(u^*), w),$$

and so $\text{fix}_\Theta(\mathbf{h}(u^*), w)$ is a g -invariant compact set. Hence, there exists a nonempty subset $\Omega \subset \text{fix}_\Theta(\mathbf{h}(u^*), w)$, that is minimal for the g -action in \mathbb{F}_Θ . By Proposition 2.3 of [12], the set Ω has to be contained in the interior of a \mathcal{S} -control set D in \mathbb{F}_Θ . Since $\Omega \subset E$ we must have that $D \subset E$ showing that E is in fact an effective \mathcal{F} -chain control set which concludes the proof. \square

As a direct consequence we have the following.

3.4 Corollary: *Any chain control set of system Σ_Θ contains a control set with nonempty interior.*

We have also that the flag type of ϕ and the \mathcal{F} -flag type of \mathcal{S} coincides.

3.5 Corollary: *With the previous assumptions, it holds that*

$$\Theta(\mathcal{F}) = \Theta(\phi).$$

Proof: Since for any two given subsets $\Theta_1, \Theta_2 \subset \Lambda$ we have that $\Theta_1 = \Theta_2$ if and only if $\mathcal{W}_{\Theta_1} = \mathcal{W}_{\Theta_2}$ it is enough for us to show that $\mathcal{W}_{\mathcal{F}}(\mathcal{S}) = \mathcal{W}_{\Theta(\phi)}$.

By Theorem 3.3 we have that the set of the chain control sets of the induced systems in \mathbb{F} and the set of the \mathcal{F} -chain control sets in \mathbb{F} coincides. Consequently

$$|\mathcal{W}_{\mathcal{F}}(\mathcal{S}) \setminus \mathcal{W}| = |\mathcal{W}_{\Theta(\phi)} \setminus \mathcal{W}| \text{ implying that } |\mathcal{W}_{\mathcal{F}}(\mathcal{S})| = |\mathcal{W}_{\Theta(\phi)}|.$$

Then it is enough to show that $\mathcal{W}_{\mathcal{F}}(\mathcal{S}) \subset \mathcal{W}_{\Theta(\phi)}$.

Since $\pi_2(\mathcal{M}(1))$ is \mathcal{S} -invariant, it contains the only \mathcal{S} -invariant control set in \mathbb{F} , and so $\pi_2(\mathcal{M}(1)) = E(1)$. Let then $w \in \mathcal{W}_{\mathcal{F}}(\mathcal{S})$. By definition $E(w)$ is the only \mathcal{F} -chain control set containing $D(w)$. Moreover, since $w \in \mathcal{W}_{\mathcal{F}}(\mathcal{S})$ we obtain $E(w) = E(1) = \pi_2(\mathcal{M}(1))$.

Consequently

$$D(w) \subset \bigcup_{u \in \mathcal{U}} \text{fix}(\mathbf{h}(u), 1).$$

Since any other choice of positive Weyl chamber just conjugates the flag types, we can assume w.l.o.g. that

$$\text{int } \mathcal{S} \cap \exp \mathfrak{a}^+ \neq \emptyset.$$

Then $wb_0 \in D(w)_0$ and there exists $u \in \mathcal{U}$ such that $wb_0 \in \text{fix}(\mathbf{h}(u), 1)$, which gives us that $wb_0 = kb_0$ where $k \in K$ is such that $\text{Ad}(k)H_\phi = \mathbf{h}(u)$. The equality $wb_0 = kb_0$ implies that $wH_\phi = \text{Ad}(k)H_\phi = \mathbf{h}(u)$ and consequently

$$\text{fix}(\mathbf{h}(u), 1) = \text{fix}(wH_\phi, 1) = (wZ_{H_\phi}w^{-1})wb_0 = \text{fix}(wH_\phi, w) = \text{fix}(\mathbf{h}(u), w).$$

Hence $w \in \mathcal{W}_{\Theta(\phi)}$ implying that $\mathcal{W}_{\mathcal{F}}(\mathcal{S}) \subset \mathcal{W}_{\Theta(\phi)}$ and concluding the proof. \square

The above corollary implies in particular that $\Theta(\mathcal{S}) \subset \Theta(\phi)$. We are now interested to see what happens when the equality holds, that is, when any chain control set of Σ_Θ contains exactly one control set with nonempty interior.

3.6 Definition: Let $\Theta \subset \Lambda$ and $w \in \mathcal{W}$. We say that the chain control set $E_\Theta(w)$ is **(uniformly) hyperbolic** if for each $(u, x) \in \mathcal{M}_\Theta(w)$ there exists a decomposition

$$T_x \mathbb{F}_\Theta = E_{u,x}^- \oplus E_{u,x}^+$$

such that the following properties hold:

(H1) $(d\varphi_{t,u})_x E_{u,x}^\pm = E_{\phi_t(u,x)}^\pm$ for all $t \in \mathbb{R}$ and $(u, x) \in \mathcal{M}_\Theta(w)$.

(H2) There exist constants $c \in (0, 1]$, $\lambda > 0$ such that for all $(u, x) \in \mathcal{M}_\Theta(w)$ we have

$$|(d\varphi_{t,u})_x v| \leq c^{-1} e^{-\lambda t} |v| \quad \text{for all } t \geq 0, v \in E_{u,x}^-,$$

and

$$|(d\varphi_{t,u})_x v| \geq c e^{\lambda t} |v| \quad \text{for all } t \geq 0, v \in E_{u,x}^+.$$

(H3) The linear subspaces $E_{u,x}^\pm$ depend continuously on (u, x) , i.e., the projections $\pi_{u,x}^\pm : T_x \mathbb{F}_\Theta \rightarrow E_{u,x}^\pm$ along $E_{u,x}^\mp$ depend continuously on (u, x) .

In [5] it is shown that any chain control set $E_\Theta(w)$ is partially hyperbolic, that is, the above decomposition of the tangent spaces have a third subspace that corresponds to the tangent space of the submanifold $\text{fix}_\Theta(h(u), w)$. By Theorem 5.6 of [5], hyperbolicity happens if and only if $\langle \Theta(\phi) \rangle \subset w\langle \Theta \rangle$ if and only if for any $u \in \mathcal{U}$ the set $\text{fix}_\Theta(h(u), w)$ consists of only one point. Consequently, if $E_\Theta(w)$ is hyperbolic, Theorem 6.1 of [5] implies that $\text{cl}(D_\Theta(w)) = E_\Theta(w)$.

The above together with the assumption that the flag types agree give us the following result concerning invariant control sets.

3.7 Proposition: *If $\Theta(\mathcal{S}) = \Theta(\phi)$ then*

$$D_\Theta(1) = E_\Theta(1) \quad \text{and} \quad \text{cl}(D_\Theta(w_0)) = E_\Theta(w_0)$$

for any $\Theta \subset \Lambda$.

Proof: We will show only the first equality since the other is obtained from it by considering the negative time systems. By the above discussion we have that and Theorem 3.3 we have that the chain control set $E_{\Theta(\phi)}(1) \subset \mathbb{F}_{\Theta(\phi)}$ is uniformly hyperbolic and so it is the closure of a control set. Therefore,

$$E_{\Theta(\phi)}(1) = \text{cl}(D_{\Theta(\phi)}(1)) = D_{\Theta(\phi)}(1),$$

since $D_{\Theta(\phi)}(1)$ is closed. Moreover, Theorem 4.3 of [12] assures that $\pi_{\Theta(\mathcal{S})}^{-1}(D_{\Theta(\mathcal{S})}(1)) = D(1)$ and Proposition 8.12 of [2], together with the relation between the chain control sets and the Morse sets of the control flow, implies that $\pi_{\Theta(\phi)}^{-1}(E_{\Theta(\phi)}(1)) = E(1)$, where $\pi_{\Theta(\mathcal{S})} : \mathbb{F} \rightarrow \mathbb{F}_{\Theta(\mathcal{S})}$ and $\pi_{\Theta(\phi)} : \mathbb{F} \rightarrow \mathbb{F}_{\Theta(\phi)}$ are the canonical projections. Therefore, if $\Theta(\mathcal{S}) = \Theta(\phi)$ we have that

$$E(1) = \pi_{\Theta(\phi)}^{-1}(E_{\Theta(\phi)}(1)) = \pi_{\Theta(\mathcal{S})}^{-1}(D_{\Theta(\mathcal{S})}(1)) = D(1).$$

Since for any $\Theta \subset \Lambda$ we have that $D_\Theta(1) = \pi_\Theta(D(1))$ and $E_\Theta(1) = \pi_\Theta(E(1))$ the result follows. \square

For any $w \in \mathcal{W}$, the **domain of attraction** of $D(w)$ is the subset of \mathbb{F} given by

$$\mathcal{A}(D(w)) = \{x \in \mathbb{F}, gx \in D(w) \text{ for some } g \in \mathcal{S}\}.$$

The following lemma relates the domain of attraction and the core of an effective control set.

3.8 Lemma: *For any $w \in \mathcal{W}$ it holds that*

$$D(w)_0 = \mathcal{A}(D(w)) \cap \mathcal{A}^*(D^*(w)),$$

where $D^*(w)$ is the unique \mathcal{S}^{-1} -control set in \mathbb{F} whose core is $D^*(w)_0 = D(w)_0$ and $\mathcal{A}^*(D^*(w))$ its domain of attraction.

Proof: Since $D(w)_0 = D^*(w)_0$ we already have that

$$D(w)_0 \subset \mathcal{A}(D(w)) \cap \mathcal{A}^*(D^*(w)).$$

For any $x \in \mathcal{A}(D(w))$ there exists $g \in \mathcal{S}$ such that $gx \in D(w)$. By using condition (ii) in the definition of \mathcal{S} -control sets implies we get that $D(w)_0 \subset D(w) \subset \text{cl}(\mathcal{S}gx)$. Since $D(w)_0$ is an open set and $\text{int } \mathcal{S}$ is dense in \mathcal{S} , we have that $D(w)_0 \cap \text{int } \mathcal{S}gx \neq \emptyset$.

Analogously, if $x \in \mathcal{A}^*(D^*(w))$ we get that $D^*(w)_0 \cap \text{int } \mathcal{S}^{-1}(g'x) \neq \emptyset$ for some $g' \in \mathcal{S}^{-1}$. Therefore, if $x \in \mathcal{A}(D(w)) \cap \mathcal{A}^*(D^*(w))$ there exists $h' \in \mathcal{S}^{-1}$ and $h \in \mathcal{S}$ such that $h'g'x, hgx \in D(w)_0$, since $D(w)_0 = D^*(w)_0$. Being that in $D(w)_0$ we have controllability, there exists $k \in \mathcal{S}$ such that

$$khgx = h'g'x \Rightarrow ((h'g')^{-1}khg)x = x.$$

However, since $\text{int } \mathcal{S}$ in \mathcal{S} -invariant we get that $(h'g')^{-1}khg \in \text{int } \mathcal{S}$ showing that $x \in D(w)_0$ and concluding the proof. \square

By Theorem 6.3 of [10], the above domain of attraction can be characterized as follows: For a finite sequence $\alpha_1, \dots, \alpha_n$ in Λ let us denote by s_1, \dots, s_n the reflections with respect these roots and consider $P_i := P_{\{\alpha_i\}}$. The corresponding flag manifold is $\mathbb{F}_i = G/P_i$ and we denote by π_i the canonical projection of \mathbb{F} onto \mathbb{F}_i . Moreover, for a given subset X of \mathbb{F} we denote by $\gamma_i(X) := \pi_i^{-1}\pi_i(X)$ the operation of exhausting the subset X with the fibers of π_i .

Now, take $w \in \mathcal{W}$ and consider the reduced expressions $w = r_m \cdots r_1$ and $w_0w = s_n \cdots s_1$ with exhausting maps γ'_i and γ_j , respectively. It holds that

$$\mathcal{A}(D(w)) = \gamma_1 \cdots \gamma_n(D(w_0)) \quad \text{and} \quad \mathcal{A}^*(D^*(w)) = \gamma'_1 \cdots \gamma'_m(D(1)) \quad (2)$$

Concerning chain control sets, from Propositions 9.9 and 9.10 of [2] and the fact that $\pi_2(\mathcal{M}(w)) = E(w)$ it is straightforward to see that

$$E(w) \subset \gamma'_1 \cdots \gamma'_m(E(1)) \cap \gamma_1 \cdots \gamma_n(E(w_0)). \quad (3)$$

We can now prove the following.

3.9 Theorem: *It holds that $\Theta(\phi) = \Theta(\mathcal{S})$ if and only if*

$$\text{cl}(D_\Theta(w)) = E_\Theta(w) \quad (4)$$

for any $\Theta \subset \Lambda$ and any $w \in \mathcal{W}$.

Proof: If the equality (4) is true for any $\Theta \subset \Lambda$ we have in particular that $|\mathcal{W}_{\Theta(\mathcal{S})}| = |\mathcal{W}_{\Theta(\phi)}|$. Since $\Theta(\mathcal{S}) \subset \Theta(\phi)$ always holds, we must have $\mathcal{W}_{\Theta(\mathcal{S})} = \mathcal{W}_{\Theta(\phi)}$ and consequently $\Theta(\phi) = \Theta(\mathcal{S})$.

If reciprocally we assume that $\Theta(\phi) = \Theta(\mathcal{S})$, Proposition 3.7 implies that the closure of any invariant control is a chain control set which by Proposition 3.7 and equations (2) and (3) imply that

$$E(w) \subset \mathcal{A}^*(D^*(w)) \cap \gamma_1 \cdots \gamma_n (\text{cl}(D(w_0))) .$$

Moreover, since $\pi_i : \mathbb{F} \rightarrow \mathbb{F}_i$ is a continuous open map between compact topological spaces, we have that $\gamma_i(\text{cl}(X)) = \text{cl}(\gamma_i(X))$ for any subset X of \mathbb{F} . Therefore

$$\gamma_1 \cdots \gamma_n (\text{cl}(D(w_0))) = \text{cl}(\gamma_1 \cdots \gamma_n (D(w_0)))$$

and so

$$E(w) \subset \mathcal{A}^*(D^*(w)) \cap \text{cl}(\mathcal{A}(D(w))) .$$

Since by Lemma 3.8 it holds that $\mathcal{A}^*(D^*(w)) \cap \mathcal{A}(D(w)) = D(w)_0$ we get that

$$E(w) \subset \mathcal{A}^*(D^*(w)) \cap \text{cl}(\mathcal{A}(D(w))) \subset \text{cl}(D(w)_0) = \text{cl}(D(w)) \subset E(w)$$

showing that $\text{cl}(D(w)) = E(w)$ for any $w \in \mathcal{W}$.

Since for any $\Theta \subset \Lambda$ and any $w \in \mathcal{W}$ we have that

$$\pi_\Theta(D(w)) = D_\Theta(w) \quad \text{and} \quad \pi_\Theta(E(w)) = E_\Theta(w)$$

the result follows. \square

3.10 Remark: Let A, B are $n \times n$ -matrices with trace zero and consider on \mathbb{R}^n the bilinear system

$$\dot{x}(t) = (A + u(t)B)x(t).$$

It is well known that the behaviour of such system is intrinsically connected with its induced control-affine system on the Grassmannians spaces. Since $A, B \in \mathfrak{sl}(n)$, we have a well defined right-invariant system on the semisimple Lie group $Sl(n)$. It is well known that any Grassmanian space is a flag manifold \mathbb{F}_Θ of $Sl(n)$ for some appropriated Θ . Moreover, any system induced by the bilinear one on a Grassmanian space coincides with the system induced by the right-invariant system on the corresponding flag. Therefore, the above results can be used, for instance, in order to analyze the behaviour of bilinear systems on \mathbb{R}^n .

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